Structural approach to the modeling of a turbulent mixing layer

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This paper combines a structural approach by deriving a turbulent coherent structure—which we call an *eigenlet*—as an eigenfunction of the Navier-Stokes equations, with a new (curl-type) eddy viscosity model (which is more representative of intermediate scales than the classical Boussinesq eddy viscosity) to describe a fully developed turbulent mixing layer without using any empirical input. This result is achieved by invoking the self-consistency condition that the mean flow and the eigenlet have the same spread angle. Using self-similar variables and the modeled equations, we obtain the mean flow and the eigenlet. Several flow features, such as Reynolds stress, mixing layer spread and inclination angles, and the average structure passage frequency have been calculated; these results are in good agremeent with experimental data.

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I. INTRODUCTION

A viable theory of turbulent shear flows remains a major challenge despite numerous pursuits, particularly numerical and experimental. Studies of quasideterministic coherent structures (CS) in a number of near-wall and free turbulent shear flows have shown that CS contribute significantly to turbulent transport and, being sensitive to external forcing, permit control of turbulence phenomena such as heat and mass transfer, combustion and chemical reaction, and drag and aerodynamic noise [1]. Unfortunately, this knowledge has so far found virtually no direct use in developing or advancing theoretical tools. Lumley proposed that CS can be considered as eigenfunctions of a linear operator based on the velocity correlation tensor (determined from experiments or numerical simulation), and developed a theory using proper orthogonal decomposition (POD) [2]. The advantage of using POD lies in minimizing the number of basis functions needed to approximate a turbulent flow. This method has been applied, in particular, to generate a dynamical system of ODE's for eigenfunction amplitudes and study the bursting phenomenon in near-wall turbulence.

In our approach, we view CS as a compact eigensolution of the governing equations; we term such localized eigenfunctions as *eigenlets*. An unusual feature of the eigenlet is that it is an eigenfunction of a nonlinear boundary-value problem which decays as $t\rightarrow\pm\infty$. The eigenlet $u_{\text{en}}(t)$ is a single impulse satisfying homogeneous boundary conditions and the Navier-Stokes equations (NSE's) or some modified form of NSE that accounts for interaction of such impulses. This modification is necessary since in a viscous flow a solitonlike solution can exist only for some specific Reynolds numbers (such a modification was done in Refs. [3,4]). However, the problem may have a continuous spectrum when solutions exist for some range of Re. We show that such a situation arises in the mixing layer case, where the presence of a continuous spectrum simplifies the problem.

Note that an eigenlet is *not* a basis function. A representation of a turbulent signal, e.g., a time series of a fluctuation velocity component at a fixed point, is given by

$$u(t)=\sum_n A_n u_n \ , \tag{1}$$

where u_n are the given basis functions (usually eigenfunctions of some linear operator, e.g., harmonic functions), and A_n are random coefficients. The simplest eigenlet tion is

$$u(t)=\sum_n u_{\text{en}}(t-t_n) \ , \tag{2}$$

where u_{en} is the eigenlet centered at a random instant t_n. Equation (2) contains no amplitudes A_n because the eigenlets u_{en} are determined along with their intensities.

The flow pattern in Fig. 1 (an instantaneous snapshot of the transitional region of a mixing layer from Brown and Roshko [5]) motivates our study of a single structure in a comoving frame of reference without considering its interaction with neighboring structures. The problem of deriving such a structure in an inviscid fluid has a long history starting from Prandtl [6], and was studied numerically (Fig. 2) using discrete vortices and subsequent rediscretization [7]. Using integrodifferential equations, Sadovsky and Taganov [8] obtained an exact solution for the velocity field of a spanwise roll (Fig. 3). The maximum spiral thickness (across center) is found to have a growth rate of $0.18(U_1-U_2)$, where U_1 and U_2 are the velocities of the two streams. Such a linearly growing rolled-up structure may appear to be inadequate for describing the mixing layer, since vortical interactions

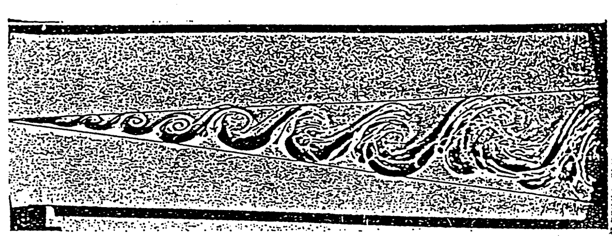

FIG. 1. Structures in mixing layer [4].

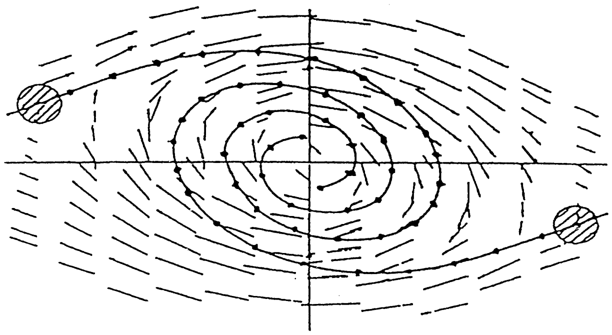

FIG. 2. Point vortex model for single structures [5].

(such as pairing) play a significant role [9,10]. These issues may be resolved using a more realistic model which explicitly accounts for these interactions. As a first step, we assume that the mixing layer spread can be determined from single growing rolled-up structures. In reality, structures evolve in a fine-scale turbulent background; note that Re also increases with x. This motivates us in this paper to consider a single structure in a turbulent medium. The turbulent background may be represented by an eddy viscosity (a nonlocal characteristic of the flow).

The self-similarity of a turbulent mixing layer follows from the fact that the problem has no inherent length scale. Therefore, any measure, e.g., thickness, should be proportional to the distance from the mixing layer origin with only one dimensionless proportionality factor—an empirical constant. This constant can be estimated from the growth rate of a single turbulent structure (in a frame fixed with the structure) if one assumes that the mixing layer also spreads at the same rate. Using the structural approach, we develop a model which is capable of calculating such an empirical constant in the mixing layer.

We begin our description of the turbulent mixing layer by proposing a curl-type model of the eddy viscosity. We assume that the mean flow and the structure (*eigenlet*) spread at the same rate, so that to maintain self-consistency the same eddy viscosity acts on both the mean flow and eigenlet. Note that this assumption will be less accurate for the classical Reynolds stress-based model, since, in the classical model, the eigenlet will constitute a larger contribution to the eddy viscosity than smaller scales; thus the eddy viscosity for the mean flow

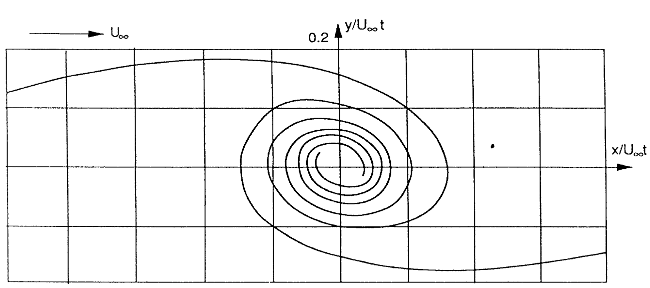

FIG. 3. Single inviscid structure [6].

in a Reynolds stress approach would have to be larger then that for the eigenlet. Since a curl-type eddy viscosity is more representative of intermediate scales, our assumption of the same eddy viscosity for the eigenlet and mean flow is reasonable. We find that our curl-type eddy viscosity gives results which are in better agreement with experimental data than the Boussinesq model.

Using this approach, the mean flow is determined, and its stability is investigated. Following this a single turbulent structure compatible with the mean flow is obtained for which several characteristics such as its mean velocity, Reynolds stress distributions, and spread angle are found; finally, we compare these results with available experimental data.

II. CURL-TYPE EDDY VISCOSITY MODEL

The simplest (classical) model due to Boussinesq introduces a scalar eddy viscosity depending, in general, on time and space coordinates and some integral flow parameters. Starting from the NSE,

$$\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\Delta v_i\ ,\quad \frac{\partial v_i}{\partial x_i}=0\ ,$$

if we put $v_i=V_i+v_i'$ and $p=P+p'$, where V_i and P are the time-averaged velocity and pressure, then the Reynolds equations for the averaged fields are

$$\frac{\partial V_i}{\partial t}+V_j\frac{\partial V_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial P}{\partial x_i}+\nu\Delta V_i-\frac{\partial\overline{v_i'v_j'}}{\partial x_j}\quad \frac{\partial V_i}{\partial x_i}=0\ . \tag{3}$$

Introducing the Boussinesq eddy viscosity ν_B,

$$2\nu_B S_{ij}=\tfrac{1}{3}\overline{v'^2}\delta_{ij}-\overline{v_i'v_j'}\ , \tag{4}$$

where $\overline{v'^2}=\overline{v_i'v_j'}$ and $S_{ij}=\frac{1}{2}[(\partial V_i/\partial x_j)+(\partial V_j/\partial x_i)]$, we obtain

$$\frac{\partial V_i}{\partial t}+V_j\frac{\partial V_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial P_e}{\partial x_i}+2\frac{\partial(\nu_{TB}S_{ij})}{\partial x_j}\ ,\quad \frac{\partial V_i}{\partial x_i}=0\ ,$$
$$P_e=P+\overline{v'^2}\ , \tag{5}$$

where ν is the molecular viscosity, ν_T is the total viscosity, and $\nu_{TB}=\nu+\nu_B$.

Thus the problem is reduced to finding an equivalent laminar viscous flow, with an effective viscosity ν_{TB}, whose velocity field is the same as the averaged turbulent one.

Relations (4) have well-known drawbacks. Conflicts arise, for example, in any parallel flow where (4) yields $v_x'^2=v_y'^2=v_z'^2$; this demand of isotropy is definitely wrong for a one-dimensional channel flow. It is well known [11] that the eddy viscosity, in principle, should be a fourth order tensor. However, a scalar eddy viscosity, being very convenient, is used even in modern semiempirical approaches.

Here we present an alternative definition of the eddy viscosity. The NSE can be written as

$$\frac{\partial \mathbf{v}}{\partial t}+\mathrm{grad}h+\boldsymbol{\omega}\times\mathbf{v}=\nu\Delta\mathbf{v}=-\nu\mathrm{curl}\boldsymbol{\omega}\ ,\quad h=p+\rho\frac{v^2}{2}\ . \tag{6}$$

Using $\boldsymbol{\omega}=\boldsymbol{\Omega}+\boldsymbol{\omega}'$, $h=H+h'$, and averaging (6) yields

$$\mathrm{grad}H+\boldsymbol{\Omega}\times\mathbf{V}=-\nu\mathrm{curl}\boldsymbol{\Omega}-\overline{\boldsymbol{\omega}'\times\mathbf{v}'}\ . \tag{7}$$

Now we propose that

$$\overline{\boldsymbol{\omega}'\times\mathbf{v}'}=\nu_\omega\mathrm{curl}\boldsymbol{\Omega}\ , \tag{8}$$

where ν_ω is a vorticity-based eddy viscosity; we may give it a simple name: vortiscosity. This relationship is similar to that of Taylor's theory of vorticity transport [12], but the left hand side of (8) is not the same as $\mathrm{div}(\overline{v_i'v_j'})$ used in Taylor's theory. Note that (8), containing a correlation between two vectors, is frame invariant.

Applying the *curl* operator on each term of (7), the equation for averaged vorticity becomes

$$V_j\frac{\partial\Omega_i}{\partial x_j}-\Omega_j\frac{\partial \mathbf{V}_i}{\partial x_j}=\frac{\partial}{\partial x_j}\left[\nu_{T\omega}\left[\frac{\partial\Omega_i}{\partial x_j}-\frac{\partial\Omega_j}{\partial x_i}\right]\right]\ , \tag{9}$$

where $\nu_{T\omega}=\nu+\nu_\omega$. For the more general case of an unsteady large-scale motion with a turbulent background, we assume that Eq. (7) can be generalized to

$$\frac{\partial V_i}{\partial t}+V_j\frac{\partial V_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial P}{\partial x_i}+\nu_{T\omega}\Delta V_i\ ,\quad \frac{\partial V_i}{\partial x_i}=0\ . \tag{10}$$

Introduction of vorticity-based eddy viscosity gives just three scalar relations from (8), unlike the six relations obtained from (4). When $\nu_\omega=$const, (8) produces a condition, namely $\mathrm{div}(\boldsymbol{\omega}'\times\mathbf{v}')=0$, which is not satisfied in general and is a limitation of this model.

Physically, the difference between models (4) and (8) is as follows. First of all, it is apparent that in general, while ν_B is dominated by large scales, ν_ω is dominated by intermediate ones. In incompressible viscous flows without body forces, momentum and vorticity cannot be generated inside the flow domain; vorticity can be produced only at the walls. This statement for a time-averaged velocity field is, generally speaking, incorrect because of mean momentum transfer by turbulent fluctuations. Equation (4) suggests that the generation of mean momentum by the eddy viscosity is not possible, while model (8) disallows such a generation for vorticity fields. This statement follows clearly from (5) and (9). Mathematically, the difference between (5) and (10) is that in (10) the variable eddy viscosity $\nu_{T\omega}$ appears *outside* the Laplace operator in contrast to (5). Comparison of (5) with (10) gives ν_{TB}, if $\nu_{T\omega}$ and V_i are known. If ν_{TB} is known, then Reynolds stresses can be found using (4). We get

$$2\frac{\partial(\nu_{TB}S_{ij})}{\partial x_j}=\nu_{T\omega}\Delta V_i\ ,$$

from which we obtain

$$2\frac{\partial\nu_{TB}}{\partial x_j}S_{ij}+(\nu_{TB}-\nu_{T\omega})\Delta V_i=0\ . \tag{11}$$

If $\nu_{TB}=$const, then $\nu_{TB}=\nu_{T\omega}$, which is generally not valid. Note that (11) is equivalent to a system of equations for only one variable ν_{TB}, and these equations can be incompatible. To avoid this difficulty, (11) is used only for the principal velocity component $V_i=U_x$.

Thus, with the help of the vortiscosity model, the problem for turbulent flows is reduced to a modified form (10) of NSE's where viscosity (in general, a function of space coordinates and time) is placed *before* the Laplace operator. Now we use this model to describe an averaged turbulent field by studying its linear stability and calculating an isolated large-scale vortex. For this, the well-known similarity features of turbulent motion will be utilized, which makes our analyses easier.

III. SELF-SIMILAR VARIABLES

With the spatially developing mixing layer in mind, we now use the two-dimensional (2D) version of NSE's in polar coordinates (r,ϕ) with the origin $r=0$ (see Fig. 4):

$$\frac{\partial V_r}{\partial t}+V_r\frac{\partial V_r}{\partial r}+\frac{V_\phi}{r}\frac{\partial V_r}{\partial\phi}-\frac{V_\phi^2}{r}=-\frac{1}{\rho}\frac{\partial P}{\partial r}+\nu_\omega\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V_r}{\partial r}\right]-\frac{V_r}{r^2}+\frac{\partial^2 V_r}{r^2\partial\phi^2}-\frac{2}{r^2}\frac{\partial V_\phi}{\partial\phi}\right]\ ,$$

$$\frac{\partial V_\phi}{\partial t}+V_r\frac{\partial V_\phi}{\partial r}+\frac{V_\phi}{r}\frac{\partial V_\phi}{\partial\phi}-\frac{V_rV_\phi}{r}=-\frac{1}{\rho}\frac{\partial P}{r\partial\phi}+\nu_\omega\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V_\phi}{\partial r}\right]-\frac{V_\phi}{r^2}+\frac{\partial^2 V_\phi}{r^2\partial\phi^2}-\frac{2}{r^2}\frac{\partial V_r}{\partial\phi}\right]\ , \tag{12}$$

$$\frac{\partial(rV_r)}{\partial r}+\frac{\partial V_\phi}{\partial\phi}=0\ .$$

Note that here we neglect molecular viscosity. In the idealized mixing layer when the molecular viscosity is omitted, the difference $U_0=U_1-U_2$ is the only velocity scale. The absence of a length scale in the problem leads

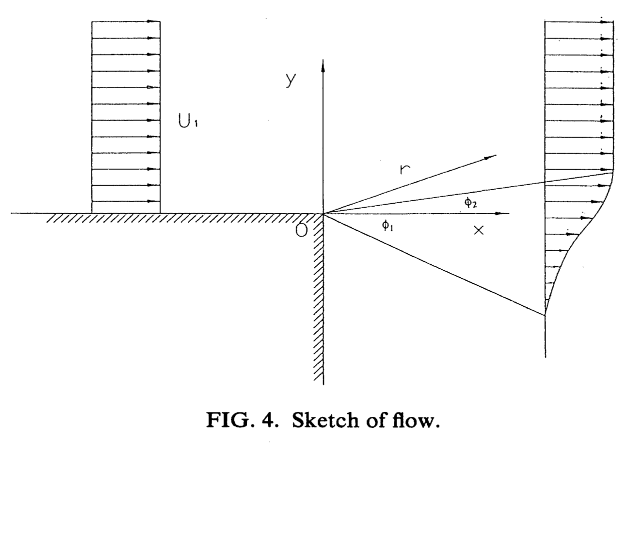

FIG. 4. Sketch of flow.

to a similarity conjecture. In particular, the eddy viscosity, with dimension of that of velocity times length, must be proportional to the distance from the origin, i.e.,

$$\nu_\omega = \gamma U_0 r f(\phi) \ , \tag{13}$$

where $f(\phi)$ describes the angular distribution of the eddy viscosity that will be specified, and the dimensionless coefficient γ serves as an inverse turbulent Reynolds number: $\mathrm{Re}=1/\gamma$. Now we introduce the dimensionless variables

$$\xi = \ln\left[\frac{r}{r_0}\right] , \quad \tau = \frac{U_0 t}{r} , \quad u = \frac{V_r}{U_0} ,$$
$$v = \frac{V_\phi}{U_0} , \quad q = \frac{P}{\rho U_0^2} . \tag{14}$$

Noting that

$$\frac{\partial}{\partial t} = \frac{U_0}{r}\frac{\partial}{\partial \tau} , \quad r\frac{\partial}{\partial r} = \frac{\partial}{\partial \xi} - \tau\frac{\partial}{\partial \tau} \equiv D ,$$

and substituting the variables in (12), we have

$$\frac{\partial u}{\partial \tau} + uDu + v\frac{\partial u}{\partial \phi} - v^2 = -Dq + \gamma f\left[D^2 u - u + \frac{\partial^2 u}{\partial \phi^2} - 2\frac{\partial v}{\partial \phi}\right] ,$$

$$\frac{\partial v}{\partial \tau} + uDv + v\frac{\partial v}{\partial \phi} - uv = -\frac{\partial q}{\partial \phi} + \gamma f\left[D^2 v - v + \frac{\partial^2 v}{\partial \phi^2} + 2\frac{\partial u}{\partial \phi}\right] , \tag{15}$$

$$Du + u + \frac{\partial v}{\partial \phi} = 0 .$$

We will use the spanwise vorticity $\Omega_z = (1/r)([\partial(rV_\phi)/\partial r] - (\partial V_r/\partial\phi))$ transformed to the nondimensional function

$$\Omega = \frac{r\Omega_z}{U_0} = \frac{\partial v}{\partial \xi} + v - \frac{\partial u}{\partial \phi} . \tag{16}$$

Eliminating the pressure terms from (15), the dimensionless equation for vorticity becomes

$$\frac{\partial \Omega}{\partial t} + u(D\Omega - \Omega) + v\frac{\partial \Omega}{\partial \phi} = \gamma\left[f(D^2\Omega - D\Omega) + \frac{\partial}{\partial \phi}\left[f\frac{\partial \Omega}{\partial \phi}\right]\right] . \tag{17}$$

Now we proceed to find solutions for the mean flow.

IV. MEAN FLOW

The mean velocity depends on angle ϕ only and not on ξ. Using the subscript zero for the steady solution and considering $D\equiv 0$ in this case, from (15) we obtain

$$v_0 u_0' - v_0^2 = \gamma f(u_0'' - u_0 - 2v_0') , \quad u_0 + v_0' = 0 , \tag{18}$$

$$q_0' = \gamma f(v_0'' - v_0 + 2u_0') . \tag{19}$$

In this section the prime denotes differentiation with respect to ϕ.

Note that Eqs. (18) for velocity components u_0 and v_0 are decoupled from (19). System (18) is of third order; therefore only *three* boundary conditions for velocity are required. If we were to use Eq. (17) for Ω along with (16) and the last equation in (15) for the basic flow, we would obtain a fourth order system which requires *four* boundary conditions. Note that while obtaining (18), pressure is considered to be independent of ξ. In general, when the velocity is independent of x, Eqs. (15) allow the additive term $C\xi$ (with $C=$const) in a solution for pressure. However, this would mean that pressure becomes infinite when $r\to\infty$; to avoid this C must be zero.

Thus, for a boundary value problem in the interval $\varphi_1 \le \phi \le \varphi_2$, we use two conditions on the high-speed boundary, $\phi=\varphi_2$, and a third condition on the low-speed boundary, $\phi=\varphi_1$. Such a choice is reasonable since the high-speed flow is given and entrainment from the low-speed side is a consequence; an entrainment velocity U_y is not known *a priori* and must be found.

In the following we study the mixing layer from a backward-facing step (Fig. 4). The boundary conditions are

$$u_0 = -1 \ ; \quad v_0 = 0 \ \text{at} \ \phi = \pi$$

and (20)

$$v_0 = 0 \quad \text{at} \ \phi = -\pi/2 ,$$

the velocity u_0 at $\phi=-\pi/2$ must be determined. Such a problem formulation (20) with a large angle is not compatible with the boundary layer approximation, and we therefore use the entire set of Eqs. (18).

We apply two models for the eddy viscosity dependence on f: (a) uniform distribution $f\equiv 1$, and (b) proportional to the averaged vorticity, $f=|\Omega_0|/|\Omega_0|_{\max}$ [because of the multiplier γ in (13), $f(\phi)$ may be normalized]. Note that case (a) assumes the eddy viscosity to be a function of coordinate only and not the velocity field (the Prandtl-Bonsinesq model), while in case (b) it depends on the mean vorticity (Prandtl model). Equations for Ω_0 that follow from (16) and (17) are

$$\Omega_0 = v_0 - u_0' = v_0'' + v_0 , \tag{21}$$

$$\gamma f \Omega_0' = v_0 \Omega_0 . \tag{22}$$

Case (a) is related to Prandtl's hypothesis that the eddy viscosity in the mixing layer is proportional to the velocity difference and the width of the mixing zone, the latter being proportional to the distance from the origin. A disadvantage of such a model is that the eddy viscosity considered does not depend on ϕ and is not zero outside the mixing zone. However, this disadvantage is not crucial since the basic flow is uniform outside the mixing zone, and, therefore, the value of the eddy viscosity there is of no consequence. When $f\equiv 1$, Eqs. (21) and (22) differ from those used by Görtler [12] (with the boundary layer approximation) only by the presence of v_0 on the right-hand side of (21). This difference in the equations disappears as $\mathrm{Re}\to\infty$; however, the results remain different even in this limiting case due to different boundary conditions. Görtler applied the condition $v_0=0$ at $\phi=0$, that made the problem symmetric with respect to ϕ. We use conditions (20), and consequently our solution is not symmetric.

Model (b) for the eddy viscosity ν_e in Prandtl's hypothesis,

$$\nu_e=l^2\left|\frac{dU_x}{dy}\right| ,$$

where U_x is the longitudinal velocity component. Tollmien [9] used this formula for the mixing layer problem with boundary layer approximation.

In our problem, mixing length 1 is proportional to r, and instead of $|dU_x/dy|$ we use $\sqrt{I_2/2}$, where I_2 is the second invariant of the velocity gradient tensor and is given by

$$I_2=4\left[\frac{\partial V_r}{\partial r}\right]^2+2\left[\frac{\partial V_\phi}{\partial r}-\frac{V_\phi}{r}+\frac{1}{r}\frac{\partial V_r}{\partial\phi}\right]^2$$
$$=2\left[\frac{U_0}{r}\right]^2\Omega_0^2 .$$

Therefore, Prandtl's formula reduces to our case (b) as follows:

$$\nu_e=l^2U_0|\Omega_0|/r . \tag{23}$$

Using $f=|\Omega_0|/|\Omega_0|_{\max}$ and $\mathrm{Re}_1=\mathrm{Re}|\Omega_0|_{\max}$ in (22), we obtain

$$|\Omega_0|\Omega_0'=\mathrm{Re}_1v_0\Omega_0 . \tag{24}$$

In the mixing layer, Ω_0 is negative, i.e., $\Omega_0=-|\Omega_0|$. Using this in (24) and assuming that Ω_0 is nonzero, (24) reduces to $\Omega_0'=-\mathrm{Re}_1v_0$, and with the help of (21), to

$$v_0'''+v_0'+\mathrm{Re}_1v_0=0 .$$

After solving for v_0, the vorticity Ω_0 has the form

$$\Omega_0=C_1\exp(\beta_1\phi)+\exp(-\beta_1\phi/2)[C_2\cos(\beta_2\phi)+C_3\sin(\beta_2\phi)] , \tag{25}$$

where β_1 is the unique real root of the equation $\beta^3+\beta+\mathrm{Re}_1=0$, and $\beta_2=(1+3\beta_1^2/4)^{1/2}$.

Because of the presence of the exponential function in (25), it is impossible to choose constants C_1, C_2, and C_3 so as to *localize* vorticity in a thin layer. For this one can combine (25) for some interval $\phi_1\le\phi\le\phi_2$ with $\Omega_0\equiv0$ [also a solution of (24) that corresponds to a uniform flow] outside the interval; values of ϕ_1 and ϕ_2 are zeros of (25). Starting from $\phi=0$ with some tentative v_0, v_0', and v_0'', and integrating (24) in the negative and positive directions up to the first zeros of $\Omega_0(\phi)$, we find ϕ_1 and ϕ_2. Then v_0, v_0' and v_0'' are calculated with the help of a shooting procedure to satisfy $u_0=\cos(\phi)$ and $v_0=-\sin(\phi_1)$ [i.e., $U_x=1$ and $U_y=0$] at $\phi=\phi_2$, and $U_x=u_0\cos(\phi)-v_0\sin(\phi)=0$ at $\phi=\phi_1$. Notice that the vortex boundaries ϕ_1 and ϕ_2 can also be found for problem (a), if one assigns some threshold (cutoff) value of Ω_0 at the boundaries (this value Ω_b is chosen as $\Omega_b=0.01\Omega_m$). The difference

$$\Delta\phi=\phi_2-\phi_1 \tag{26}$$

will be used in the following.

V. STABILITY

To study the stability of the basic flow we need to derive equations for disturbances. For this, we use the representation $u=u_0(\phi)=u_d(\phi,\xi,\tau)$; similarly for v, q, and Ω. Variables (14) make the evolution problem for (15) rather unusual because the first two equations in (15) are both second order with respect to τ. In addition, the linearized equations (15) do not allow an exponential solution in τ as the initial NSE does with respect to t. To avoid these difficulties and simplify calculations we apply the local approximation that $\Delta r/r_0$ is a small parameter, with Δr as a characteristic length scale of u_d, and r_0 as the distance from the origin. This simplification is similar to the parallel-flow approximation used in stability studies of mixing layers [10]. An advantage of our approach in comparison with the quasiparallel approximation is that we account for the nonparallel nature of the basic flow in the leading order of the expansion with respect to $\Delta r/r_0$. This allows us (i) to describe vortex distortion induced by the nonparallel nature of the flow, and (ii) to avoid a major drawback of the parallel approach which fails to satisfy the boundary conditions for the basic flow perturbations (i.e., decay of the perturbations in the transverse direction). Formally, the simplification is achieved by applying $\tau=U_0t/r_0$ [compare with (14)] and $D=\partial/\partial\xi$ in (15)–(17). For a perturbation propagating downstream with velocity C we use $\partial/\partial t=-C\partial/\partial\xi$. Then from (15) the nonlinear disturbance equations are

$$(u_0-C)\frac{\partial u_d}{\partial\xi}+v_0\frac{\partial u_d}{\partial\phi}+v_d\frac{\partial u_0}{\partial\phi}-2v_0v_d+\frac{\partial q_d}{\partial\xi}-\gamma f\left[\frac{\partial^2u_d}{\partial\xi^2}-u_d+\frac{\partial^2u_d}{\partial\phi^2}-2\frac{\partial v_d}{\partial\phi}\right]=v_d^2-u_d\frac{\partial u_d}{\partial\xi}-v_d\frac{\partial u_d}{\partial\phi} ,$$

$$(u_0-C)\frac{\partial v_d}{\partial\xi}+v_0\frac{\partial u_d}{\partial\xi}+\frac{\partial q_d}{\partial\phi}-\gamma f\left[\frac{\partial^2v_d}{\partial\xi^2}-v_d+\frac{\partial^2v_d}{\partial\phi^2}+2\frac{\partial u_d}{\partial\phi}\right]=v_d\frac{\partial u_d}{\partial\xi}-u_d\frac{\partial v_d}{\partial\xi} , \tag{27}$$

$$\frac{\partial u_d}{\partial\xi}+u_d+\frac{\partial v_d}{\partial\phi}=0 .$$

To find an eigenlet we must obtain a solution of (27) satisfying appropriate boundary conditions. We have to obtain a nontrivial solution for some (eigen)value of parameter C; parameter γ remains to be determined.

We begin with the problem of linear stability for the mean turbulent mixing layer. We reconsider the linear stability of the mixing layer with the Görtler-Blasius profile, studied in a number of works (see [14]), by applying our nonparallel approach and the new model of eddy viscosity.

To study infinitesimal disturbances we linearize (27) and neglect terms on the right-hand sides. Then the normal mode for disturbances can be applied:

$$(u_d, v_d, q_d, \Omega_d) = [u_1(\phi), v_1(\phi), q_1(\phi), \Omega_1(\phi)] \times \exp[i\alpha(\xi - C\tau)] + \text{c.c.} \qquad (28)$$

Here $C = C_r + iC_i$ is the complex phase velocity, and c.c. denotes the complex conjugate. Substitution of (28) in (27) yields

$$i\alpha(u_0 - C)u_1 + (u_0' - 2v_0)v_1 + v_0 u_1' + i\alpha q_1 = \gamma f[u_1'' - (1+\alpha^2)u_1 - 2v_1'] ,$$

$$i\alpha(u_0 - C)v_1 - i\alpha v_0 u_1 + q_1' = \gamma f[v_1'' - (1+\alpha^2)v_1 + 2u_1'] , \qquad (29)$$

$$k u_1 + v_1' = 0 , \quad k = 1 + i\alpha .$$

Differentiating the last equation, substituting v_i'' in the second equation, and using $\Omega_1 = kv_{1\_} u_1'$ due to (16), we reduce (29) to the fourth-order system

$$\gamma f \Omega_1' = v_0(\Omega_1 - kv_1) - i\alpha(u_0 - C)u_1 - (u_0' - 2v_0)v_1 - i\alpha q_1 ,$$

$$q_1' = \gamma f(i\alpha - 1)\Omega_1 + i\alpha v_0 u_1 - i\alpha(u_0 - C)v_1 , \qquad (30)$$

$$u_1' = kv_1 - \Omega_1 , \quad v_1' = -ku_1 ,$$

with boundary conditions

$$v_1 = \Omega_1 = 0 \text{ at } \phi = \pi \text{ and } \phi = -\pi/2 . \qquad (31)$$

In the case of model (b) for eddy viscosity, we assume that the disturbance velocity field is potential in the regions $-\pi \le \phi \le \phi^*$ and $\phi^{**} \le \phi \le \pi$, i.e., $\Omega_1 \equiv 0$. It then follows from (30) and (31) that

$$v_1 = C_1 \sin[k(\phi + \pi/2)] \text{ and } u_1 = -C_1 \cos[k(\phi + \pi/2)] \text{ for } -\pi/2 \le \phi \le \phi^* ,$$

$$v_1 = C_2 \sin[k(\phi - \pi)] \text{ and } u_1 = -C_2 \cos[k(\phi - \pi)] \text{ for } \phi^{**} \le \phi \le \pi .$$

Therefore, boundary conditions (31) can be replaced by

$$\Omega_1 = 0 , \quad \cos[k(\phi + \pi/2)]v_1 + \sin[k(\phi + \pi/2)]u_1 = 0 \text{ at } \phi = \phi^* ,$$

$$\Omega_1 = 0 \quad \cos[k(\phi + \pi)]v_1 + \sin[k(\phi - \pi)]u_1 = 0 \text{ at } \phi = \phi^{**} .$$

Similar inviscid conditions were used in the stability analysis of a laminar mixing layer using a quasiparallel approximation [13].

The neutral curves for model (a) are shown in Fig. 5. Usually, the mean flow scale is characterized by the local momentum thickness

$$\theta = r \int_{-\pi/2}^{\pi/2} U_x(1 - U_x)\, d\phi . \qquad (32)$$

Therefore, instead of the (turbulent) Reynolds number $\text{Re} = 1/\gamma$, we use the value of $R_\theta = r/\theta$ for the abscissa to allow comparison with experiments [15]. The limiting values of $\alpha\theta = 0.48$ and $\omega\theta/U_0 = 0.28$ shown in Fig. 5 coincide with known results [13]; however, for the experimental turbulent momentum thickness, $\alpha\theta$ and $\omega\theta/U_0$ are remarkably less than the limiting values (see section T in Fig. 5). The cross shown in Fig. 5 corresponds to the maximum value of αC_i for $R_\theta = 25.6$; correspondingly, $\omega\theta/U_0 = 0.15$. We find that the results from model (b) are quite close to those obtained by model (a).

VI. EIGENLET FOR A TURBULENT MIXING LAYER

Our stability analyses show that the mean velocity field of a turbulent mixing layer is unstable, and consequently structures will be generated in a turbulent background; this serves as a basis for obtaining a turbulent eigenlet. We know that an inviscid eigenlet exists, but the solution for finite Re is unknown. Usually, a solitonlike solution of the NSE exists only for a particular Re. The existence of a turbulent eigenlet solution for some range of Re means that the problem has a continuous spectrum. This is not surprising because (17) is of second order in τ. In the present work we prefer solving (17) in a frame moving with the structure. Thus, for the temporally developing

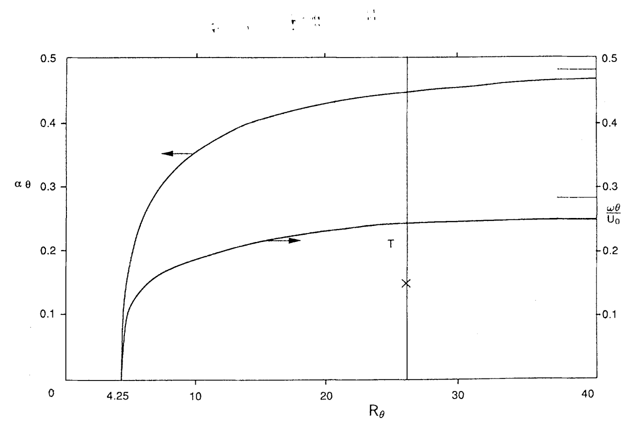

FIG. 5. Neutral curves for turbulent mixing layer.

mixing layer formed by two oppositely directed streams with velocities $\pm U_\infty$ (Fig. 3), we start from the equation for the stream function Ψ in the Cartesian frame (x,y):

$$\frac{\partial \Delta\Psi}{\partial t}+\frac{\partial \Psi}{\partial y}\frac{\partial \Delta\Psi}{\partial x}-\frac{\partial \Psi}{\partial x}\frac{\partial \Delta\Psi}{\partial y}=\nu_\omega \Delta\Delta\Psi \ . \tag{33}$$

The eddy viscosity model (13) is then modified to

$$\nu_\omega=\beta U_0^2 t \ , \tag{34}$$

where $\beta=$const; i.e., we use model (a) for eddy viscosity to simplify the analysis. This modification is needed because the flow is uniform in the streamwise direction, and the characteristic length (thickness of the mixing layer) is proportional to $U_0 t$. The relationship between β and γ will be established below.

We introduce the following transformation for dependent and independent variables:

$$\xi=\frac{x}{U_\infty t} \ , \quad \eta=\frac{y}{U_\infty t} \ , \quad \tau=\ln\frac{t}{t_0} \ , \quad \psi=\frac{\Psi}{U_\infty^2 t} \ ,$$

and use them to transform (33) to

$$\frac{\partial \Delta\psi}{\partial \tau}+\frac{\partial \psi}{\partial \eta}\frac{\partial \Delta\psi}{\partial \xi}-\frac{\partial \psi}{\partial \xi}\frac{\partial \Delta\psi}{\partial \eta}-\Delta\psi-\xi\frac{\partial \Delta\psi}{\partial \xi}-\eta\frac{\partial \Delta\psi}{\partial \eta}=\beta\Delta\Delta\psi \ , \tag{35}$$

where $\Delta=(\partial^2/\partial\xi^2)+(\partial^2/\partial\eta^2)$. A steady solution of (35) corresponds to an unsteady self-similar solution of (33) with the velocity components

$$v_x=\frac{\partial \Psi}{\partial y}=U_\infty\frac{\partial \psi}{\partial \eta} \ , \quad v_y=-\frac{\partial \Psi}{\partial x}=-U_\infty\frac{\partial \psi}{\partial \xi} \ . \tag{36}$$

It is convenient to use the following symmetric boundary conditions:

$$\frac{\partial \psi}{\partial \xi}=0 \ , \quad \frac{\partial \psi}{\partial \eta}=1 \quad \text{at } \eta=+\infty \quad \text{and} \quad \frac{\partial \psi}{\partial \xi}=0 \ , \quad \frac{\partial \psi}{\partial \eta}=-1 \quad \text{at } \eta=-\infty \ . \tag{37}$$

It is evident that the problem (35)–(37) has a solution that is independent of τ and ξ.

$$\psi_0=\int_0^\eta \mathrm{erf}(\eta/\sqrt{2\beta})d\eta \ , \tag{38}$$

which describes turbulent diffusion of an initial vortex sheet and yields the velocity distribution

$$v_x=U_\infty \mathrm{erf}\left[\frac{1}{\sqrt{2\beta}}\frac{y}{U_\infty t}\right] \ , \quad v_y=0 \ . \tag{39}$$

One can see that this solution corresponds to a tangent jump of velocity at $\beta=0$. Relation (39) is considered a trivial solution, and our goal is to find a nontrivial solution that represents a single vortex as a localized disturbance of the vortex sheet, i.e., an eigenlet.

To develop an effective algorithm for a numerical solution of the problem, we consider an integral sequence of (35). Denoting

$$\mathbf{r}=(\xi,\eta) \ , \quad \mathbf{V}=\left[\frac{\partial \psi}{\partial \eta},-\frac{\partial \psi}{\partial \xi}\right] \quad \text{and} \quad \tilde{\Omega}=\Delta\psi \ ,$$

Eq. (35) is rewritten in the form

$$\frac{\partial \tilde{\Omega}}{\partial \tau}=\beta\Delta\tilde{\Omega}-\tilde{\Omega}-\mathrm{div}[(\mathbf{V}-\mathbf{r})\tilde{\Omega}] \ . \tag{40}$$

We represent the motion as a superposition of the trivial solution and a disturbance: $\mathbf{V}=\mathbf{V}_0+\mathbf{v}$ and $\tilde{\Omega}=\tilde{\Omega}_0+\tilde{\omega}$. Then the equation for disturbances follows from (40),

$$\frac{\partial \tilde{\omega}}{\partial \tau}=\beta\Delta\tilde{\omega}-\tilde{\omega}-\mathrm{div}[(\mathbf{V}_0-\mathbf{r})\tilde{\omega}+\tilde{\Omega}_0 v+\tilde{\omega}\mathbf{v}] \ . \tag{41}$$

Integrating (41) in region S with boundary 1, we get

$$\frac{\partial \Gamma}{\partial \tau}+\Gamma=\int_1\left[\beta\frac{\partial \tilde{\omega}}{\partial n}-\mathbf{n}\cdot(\mathbf{V}_0-\mathbf{r})\tilde{\omega}+\tilde{\Omega}_0\mathbf{v}+\tilde{\omega}\mathbf{v}\right]dl \ . \tag{42}$$

For localized disturbances, the right-hand-side integral becomes zero for a sufficiently large S, and this implies that the disturbance circulation Γ decays like $\Gamma=\Gamma(0)\exp(-\tau)$. Therefore, one does not need to be concerned that an initial state has the same global circulation as the trivial solution, because a difference in disturbance circulation (Γ), if any, decays exponentially during the establishment process. The results of our calculations are given in Sec. VII.

VII. DESCRIPTION OF STRUCTURE CHARACTERISTICS

In the following, we summarize results obtained from our model. First of all, we discuss the relation between the spatial and temporal problems, which are assumed to contain the same eigenlet. Thus results to be obtained for a spatial mixing layer are directly applicable to the temporal case, with eddy viscosity (34). We use a frame moving with a velocity

$$U=\tfrac{1}{2}(U_1+U_2) \ . \tag{43}$$

In this frame, the relative flow velocity is $U_1-U=\frac{1}{2}(U_1-U_2)=U_\infty$ at $y=\infty$, and $U_2-U=-\frac{1}{2}(U_1-U_2)=-U_\infty$ at $y=-\infty$. Assuming in (13) that $rf(0)=x=Ut$, we have

$$v=\gamma U_0 U t=\gamma U_0^2\frac{U}{U_0}t=\tfrac{1}{2}\left[\gamma U_0^2\frac{1+m}{1-m}\right]t \ , \tag{44}$$

where $m=(U_2/U_1)$. Comparison of (44) and (34) gives

$$\beta=\tfrac{1}{2}\left[\gamma\frac{1+m}{1-m}\right] \ . \tag{45}$$

Let δ be the characteristic vortex size growing according to

$$\delta=\sigma U_\infty t=\sigma\frac{U_\infty}{U}x=\sigma\frac{1-m}{1+m}x \ , \tag{46}$$

where $\sigma=$const.

Notice that Eq. (45) expresses eddy viscosity dependence on m in full accordance with experimental observation [16], if we assume that β is independent of m. In the present work, we consider only the case where $m=0$. From Sadovsky and Taganov [8], for an inviscid single vortex structure, we find that $\sigma=0.36$. The angle

$$\Delta\phi_0=2\arctan(\sigma/2)\ , \tag{47}$$

corresponding to $\sigma=0.36$, is $\Delta\phi_0=0.356$.

Knowing $\Delta\phi_0$, we can estimate the eddy viscosity of the mean flow if one assumes that $\Delta\phi_0=\Delta\phi$ from (26). For an inviscid flow the angle $\Delta\phi_0$ is finite and the boundaries are sharp; therefore it is natural to use model (b) considered in Sec. IV.

Calculations show that the value $\Delta\phi_0=0.356$ is achieved at $\phi_2=0.1154$, $\phi_1=-0.2406$, $\Omega_{0\,\mathrm{min}}=-4.45$, and $\mathrm{Re}=617$. To compare these results with experimental data, we apply (23). Tollmien used $l=cx$ and recommended the experimental value $c=0.0246$. Comparing (23) with (13) we obtain $\gamma=c^2|\Omega_0|_{\mathrm{max}}$. As $\gamma=\mathrm{Re}^{-1}$, we find $c=0.0191$. We can see that the experimental value of the eddy viscosity exceeds the one obtained with an inviscid approximation. Since it is more convenient to use model (a) with $f\equiv 1$, we calculate the turbulent structure corresponding to this model. The self-consistency condition for the temporal and spatial models requires matching of two parameters: $\gamma=2\beta$ and $\Delta\phi=2\arctan(\sigma/2)$, from (44) and geometry. The boundaries of the vortex which determine $\Delta\phi$ and σ are not distinct for model (a). They were determined using the cutoff (threshold) value of $\Omega_b=0.01\Omega_{\mathrm{max}}$, so that we use Ω_b to denote the value of Ω on the boundary of the vortex. Note that the boundary of the vorticity region in the mean flow is also determined using the same cutoff. To solve the whole problem we perform the following operations: (i) start with a trial value of γ in (22); (ii) find $\Delta\phi=\phi_2-\phi_1$; (iii) find β according to (45); (iv) solve problem (35); and (v) find σ from the structure growth rate and $\Delta\phi_0$ according to (47). The problem is solved if we can find the γ for which the equality $\Delta\phi=\Delta\phi_0$ is satisfied.

The numerical problem for (35) is studied in the rectangle $|\eta|\le L$, $|\xi|\le 10L$, for which the boundary conditions corresponding to the trivial solution are satisfied. L is chosen to be large enough so that the maximum vorticity is independent of L. We begin our computations from an initial field corresponding to a prediction based on the results of the inviscid theory, use a time step $\Delta t=0.01$, and terminate calculations when the solution attains some asymptotic limit (after 1700 steps); this constitutes the first run. The calculations are then repeated, starting from initial distributions with larger and smaller disturbance amplitudes. It is found that in both cases the amplitude converges to the value obtained from the first run.

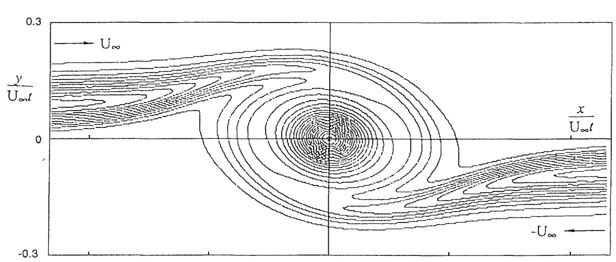

FIG. 6. Single turbulent structure (eigenlet).

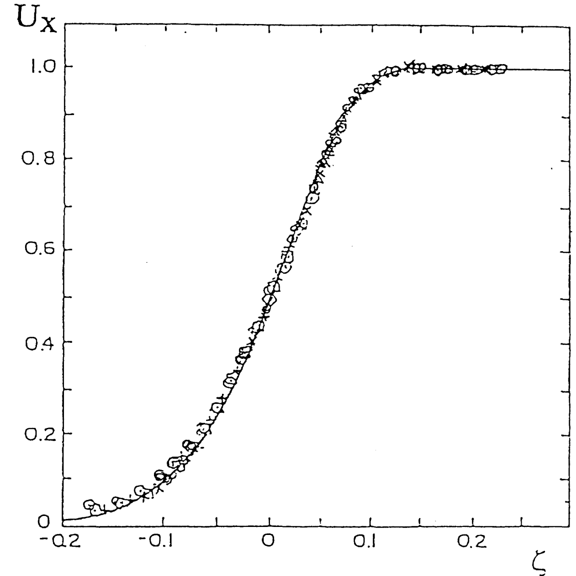

FIG. 7. Mean longitudinal velocity profile in the turbulent mixing layer.

The vortex structure obtained is shown in Fig. 6, where the isovorticity line as are depicted. The flow pattern is in qualitative agreement with the inviscid solution (Fig. 3). The main difference is the smooth distribution of the vorticity in the eddy viscosity model. The structure has the following parameters: $|\tilde{\Omega}|_{\mathrm{max}}=12.12$, $\beta=0.001\,002$, $\gamma=0.002\,004$, $\mathrm{Re}=499$, $\sigma=0.472$, and, according to (47), $\Delta\phi_0=0.4364$. The solution using model (a) (see Sec. IV) is shown in Fig. 7 with experimental data [15] obtained in a turbulent mixing layer. The independent variable ζ used here is standard for representation of mixing layer experimental data: $\zeta=(y-y_{0.5})/x$, where $y_{0.5}$ is the transverse location where $U_x=0.5$. $U_y(\zeta)$ is depicted in Fig. 8. Although detailed $U_y(\zeta)$ profile data are unavailable experimentally for the turbulent mixing layer, this profile is in qualitative agreement with data obtained in our laboratory [17]. In particular, $U_y(-\infty)$ (the entrainment velocity) value is found to be 0.038, which is close to the experimental value of 0.032. Such results

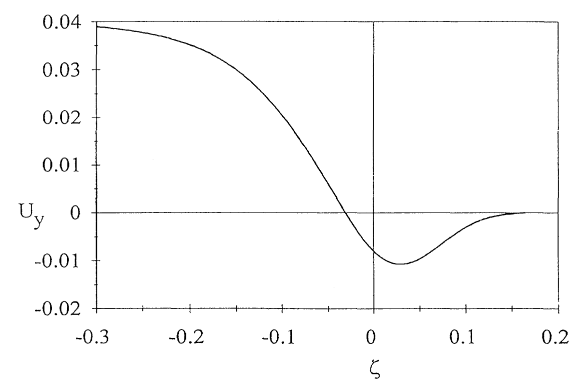

FIG. 8. Mean transverse velocity profile.

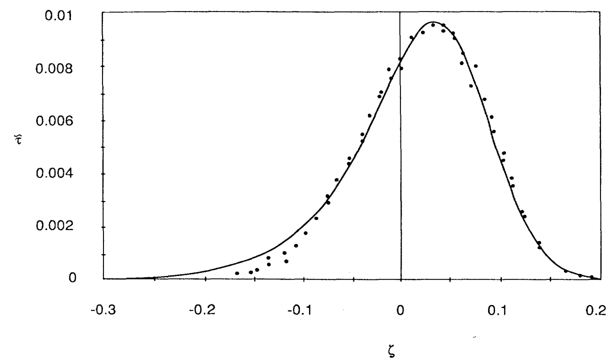

FIG. 9. Reynolds stress profile.

cannot be achieved by the boundary layer approximation, which gives a zero value. Our calculations provide a momentum thickness $\theta=0.0391r$ which coincides with the experimental value [15]; the corresponding Reynolds number is $R_\theta=25.6$. Another characteristic of a mixing layer is $\theta_{0.1}$, the momentum thickness when integration in (21) and (22) starts from the angle ϕ where $U_x=0.1$. We have $\theta_{0.1}=0.0343r$, while the experimental value is $0.035r$. Yet another characteristic of a mixing layer is its inclination angle $\phi_{0.5}$ corresponding to $U_x=0.5$. In Görtler theory, due to symmetry $\phi_{0.5}=0$. Our calculations give $\phi_{0.5}=-0.0306$, whereas the experimental values vary [18,19] from $\phi_{0.5}=-0.0473$ to $\phi_{0.5}=-0.027$, respectively.

Reynolds stresses are found according to (4),

$$\tilde{\tau}=-\overline{v_x'v_y'}=\nu_B\left[\frac{\partial V_x}{\partial y}+\frac{\partial V_y}{\partial x}\right],$$

and calculated using (11). Comparison with experimental data [17] is shown in Fig. 9. Note that the use of boundary layer approximation, $\tilde{\tau}=\nu_B\partial v_x/\partial y$, overestimates $\tilde{\tau}_{\max}$ by 20%. In addition to the mean flow, we estimate the characteristic structure passage frequency f_m in the mixing layer using results of stability analysis given in Sec. V. Assuming that f_m corresponds to perturbations with maximum αC_i, we obtain $\omega_m\theta/U_0=0.15$, where ω_m is the dimensional angular frequency. Using $\omega_m=2\pi f_m$, we find a Strouhal number $\mathrm{St}=f_m\theta/U_0$ value of 0.0239, while the experimental value [20] is 0.024. Since θ increases linearly with distance r, f_m is inversely proportional to r according to $\omega_m=\alpha C_r U_0/r$, which follows from Eq. (28). Observations [20] confirm these considerations thus justifying our stability approach.

Despite considering an ideal mixing layer ignoring boundary layers on the exit walls, and making other simplifying assumptions, our theoretical model provides results in satisfactory agreement with experimental data for fully developed turbulent mixing layers.

VIII. CONCLUDING REMARKS

We have demonstrated the importance of structures in a turbulent flow (a turbulent mixing layer) and showed that the knowledge about a single structure (eigenlet) alone allows us to determine mean characteristics of the entire flow without any empirical input. Eigenlets, being localized hydrodynamic structures, bear some resemblance to elementary particles of physics. The latter conserve their shape during free motion and could be distorted due to interactions, possibly giving rise to new particles. Also, a large-scale structure (represented by an eigenlet) can change its shape and undergo a similarity transformation, for instance. Growing mixing layer vortices are an example of such a transformation.

In applied hydrodynamics, various turbulence models have been developed containing empirical constants which unfortunately are not universal. The present work attempts to obtain such a constant for a turbulent mixing layer using a structural approach.

The nonlinear winding process plays a dominant role in structure evolution, causing a linear growth of the scale, and is followed by strong interaction between structures (such as pairings). Subsequently, small-scale turbulence arises which modifies viscosity and produces secondary structures by instability. In this paper, we account for structure interaction indirectly by introducing an eddy viscosity ν_ω. We therefore arrive at the theoretical model for the calculation of a single structure in a turbulent medium. ν_ω is found by imposing the self-consistency condition that the mean flow field and the structure have the save properties, such as eddy viscosity and spread angle. To realize this, first we have proposed a curl-type eddy viscosity model, and provided a solution to the mean flow, using self-similar variables. Then a linear stability approach is developed using self-similar variables. Following this a single turbulent structure in the mixing layer (an eigenlet) is calculated, and the mean flow field, Reynolds stress distribution, and structure passage frequency are determined. We find these results to be in good agreement with experimental data.

Although we have demonstrated our results by combining turbulence modeling and structural approaches for a turbulent mixing layer, we emphasize that such an approach may be extended to large Re flows including potential or vortical outer flows, boundary layers, wakes, and jets. The idea of self-consistency is likely also to be useful for *subgrid turbulence modeling*. The appearance of eddy viscosity is typically related with some cutoff (which is necessary for any numerical calculation) of small scales. Let us assume that we start with NSE (10) where $\nu_{t\omega}=\nu$, and also that we are able to estimate the eddy viscosity field $\nu_{T\omega}$ corresponding to the chosen cutoff (using, for example, Kolmogorov's formula or its generalization). Then, we can use (10) (or some modification of it) and repeat the calculations with the viscosity field $\nu_{T\omega}$. We can then repeat the cutoff operation and continue the process until we obtain the same eddy viscosity field as that estimated. If we succeed, the self-consistency condition is satisfied. In fact, we used such a procedure in this paper for the mixing layer steady flow.

The mixing layer case is simple in the sense that the turbulence modeling needs only one constant. In order to apply the structural approach in more complicated flows, where more constants may be involved, we may use another turbulence model, e.g., k-ε, to find the corresponding dominant eigenlet. In the classical sense the mixing layer theory is complete since we have determined and characterized the mean flow field without involving any empirical input. However, for more detailed features of this flow, such as description of vortex pairing and tearing, 3D vortex dynamics, rib formation, and vortex reconnections, more sophisticated techniques would be needed. One such technique is the structural-statistical approach, which has been used for ODE's displaying chaotic behavior [3,4], and can, in principle, be extended to turbulent flows.

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